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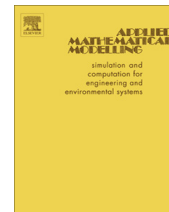
A meshless like method for the approximate solution of integral equations over polygons

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ABSTRACT

In this paper, a numerical scheme based on a Gauss-like cubature formula from Sammariva and Vianello (2007) [1] is introduced for approximate solution of integral equations over a polygonal domain with a piecewise straight lines boundary in $\mathbb{R}^2$. The proposed technique is a meshless like method with sufficient precision, which does not require any discretization of the polygon domain or any preprocessing such as mesh refinement. The error analysis of the method is provided and some numerical experiments are also presented to evaluate the performance of the proposed algorithm.

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1. Introduction

We consider the Hammerstein integral equation of the second kind over a polygon domain Ω in plane $\mathbb{R}^2$ as the form

$$u(x,y) = f(x,y) + \int\int_\Omega k(x,y,s,t)G(s,t,u(s,t))\,dsdt, \qquad (1.1)$$

where the continuous given functions f and k are defined respectively on R and $R \times R$ in which the rectangular region R enclosing the polygon Ω.

Throughout the paper, we assume that the Eq. (1.1) is uniquely solvable with u as an unknown function which should be determined and the function G satisfies in the following conditions:

(i) $G(s,t,u(s,t)) \in C(R \times (-\infty,\infty))$,
(ii) The partial derivative of G with respect to the variable u exists and satisfies in the Lipschitz condition

$$|G_u(s,t,u_1(s,t)) - G_u(s,t,u_2(s,t))| \leqslant c|u_1(s,t) - u_2(s,t)|.$$

Multivariate integral equations as (1.1) have many important applications in science and technology, e.g., the radiosity equations which is a global illumination algorithm used in 3D computer graphics [2,3]. Another form of this equations may arise in boundary integral equations in three dimensions which is defined on surface in space [4,5]. We refer the reader for a brief bibliography of such applications to the survey [6].

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There are some theoretical and numerical studies on Fredholm integral equations of the second kind (1.1). Most of the numerical methods are described by subdividing the domain Ω into triangular or quadrilateral elements, and then exerting a discretization based on approximation of unknown function by a piecewise polynomial over segments of the surface or applying a cubature over each element and finally summing up the values obtained and achieving the function values in nodal points. Generally, this discretization leads to very large linear systems which increases the complexity of the problem (see e.g., [4]).

On the other hand, a variety of numerical schemes based on the choice of cubature and subdivision type have been proposed by several authors [4,7,5,8,16]. Generally, triangulations and quadrangulations are from extensive subdivisions which may be used in adaptive algorithms. Thus, there are two problems to consider in approximating the integral in (1.1), one is choice of a suitable cubature, and the other is the selection of a domain subdivision technique which is a major difficulty. Using methods that do not require subdivision of the polygon domain can eliminate the second problem.

In recent years, meshless like methods have gained the attention of many researchers. A number of cubature formulas for a variety of regions that do not require a subdivision of the integration domain have been given in [9,10,1]. Sommariva and Vianello in [9], have been solved the problem for polygons by resorting to Radial Basis Functions (RBF) interpolation in connection with Green's integral formula, and then the method has been extended to meshless cubature from scattered data over the disk in [10]. Recently, they have introduced a completely different approach, in the special case of cubatures over polygons, which is a Gauss-like numerical integration with the Green's formula together the univariate Gauss–Legendre discretization [1]. This cubature formula dose not require subdivision and it only needs the polygon boundary in the form of a counterclockwise sequence of vertices. The formula is exact for at most $2n - 1$ degree polynomials with n arbitrary over convex, nonconvex, or even multiply connected polygons. Most recently, Li and Dagnino in [11] have constructed an adaptive numerical integration algorithm for polygons by cubatures based on quadrilateral elements and compared their results with those obtained by cubatures based on triangulation and Gauss-like method. The results demonstrate that for smooth functions, the errors for Gauss-like cubature are better than those by adaptive algorithms and equal subdivision methods. However, as the Gauss-like cubature nodes fall outside the non-convex test domain, therefore the integrand function has to be computed also in the rectangular domain containing the polygon, as mentioned in Remark 4 of [1].

The main aim of this paper is to construct a Nystrom type method based on this new Gauss-like cubature for approximate solution of the Hammerstein integral equations over a polygon in $N \sim mn^2$ nodes, with m being the number of sides of the polygon that is not orthogonal to a given line, and not lying on it. In general, these nodes spread over into the smallest rectangular regions included in the polygon. It is notable that under the present conditions, specially for polygons which one of the sides located on x-axes, the discretization nodes fall inside the polygon. However, in the presented test problems, we have considered the polygons with different points distribution whose nodes may be placed inside or outside the integration region.

The paper is organized as follows: a Nystrom type method based on Gauss-like cubature [1] for solving the Hammerstein integral equations over a polygon is introduced in Section 2. Section 3, contains an error analysis of the proposed procedure and finally in Section 4, the numerical treatments of several test problems of linear and nonlinear integral equations over some polygons with different points distribution are reported in order to efficiency and effectiveness of the method.

2. The proposed method based on Gauss-like cubature

The integral Eq. (1.1), may be written in the following abstract form:

$$u - KGu = f, \qquad (2.1)$$

where K can be considered as a nonlinear integral operator:

$$(KGu)(x,y) = \int\int_\Omega k(x,y,s,t)G(s,t,u(s,t))\,ds\,dt, \quad (x,y) \in R, \quad u \in C(R).$$

Generally, the computational issues of this class of nonlinear integral equations have two major aspects. Discretization of the Eq. (2.1) by replacing it with a sequence of finite dimensional approximating problem as

$$u_n - K_nGu_n = f, \quad n \to \infty \qquad (2.2)$$

and then solving the resulting finite system of equations for deriving the approximate solutions of (1.1), in nodal points.

For describing the key idea, we state the following lemma from [1] which is the basis of the proposed method:

Lemma 1 (*From [1]*). *Let Ω be the closure of a bounded and simply connected polygon with boundary described counterclockwise by the sequence of vertices*

$$v_i = (\alpha_i, \beta_i), \quad i = 1, \dots, l, \quad l \geqslant 3$$

and suppose $f \in C(R)$ and let α be fixed, where

$$\Omega \subseteq R = [a,b] \times [c,d], \quad \alpha \in [a,b].$$

Then, the following cubature formula is exact over Ω for all bivariate polynomials of degree at most $2n-1$,

$$I_{2n-1}(f) = \sum_{i \in \mathcal{I}_{\Omega,\alpha}} \sum_{j=1}^{n_i} \sum_{k=1}^{n} w_{ijk} f(\xi_{ijk}, \eta_{ij}),$$

where

$$\mathcal{I}_{\Omega,\alpha} = \{i : \Delta\beta_i \neq 0\} \cap \{i : \alpha_i \neq \alpha \text{ or } \alpha_{i+1} \neq \alpha\}$$

and

$$n_i = \begin{cases} n, & \Delta\alpha_i = 0, \\ n+1, & \Delta\alpha_i \neq 0. \end{cases}$$

The nodes are given by

$$\begin{cases} \xi_{ijk} = \frac{x_i(\tau_j^{n_i})-\alpha}{2}\tau_k^n + \frac{x_i(\tau_j^{n_i})+\alpha}{2}, \\ \eta_{ij} = y_i(\tau_j^{n_i}), \end{cases} \tag{2.3}$$

in which

$$x_i(t) = \frac{\Delta\alpha_i}{2}t + \frac{\alpha_i + \alpha_{i+1}}{2}, \quad y_i(t) = \frac{\Delta\beta_i}{2}t + \frac{\beta_i + \beta_{i+1}}{2}$$

and the weights are obtained as

$$w_{ijk} = \frac{1}{4}\Delta\beta_i(x_i(\tau_j^{n_i}) - \alpha)\lambda_j^{n_i}\lambda_k^n, \tag{2.4}$$

where $\{\tau_j^s\}$ and $\{\lambda_j^s\}$ for $1 \leqslant j \leqslant s$ be respectively, the nodes and the weights of the univariate Gauss–Legendre quadrature formula of degree of exactness $2s-1$ on $[-1,1]$ and Δ denoting the forward difference operator.

In order to approximate the solution of (1.1), we construct a Nystrom type method. To do so, we discretize the integral that appears in the equation by the relation based on the Gauss-like cubature rule of degree of exactness $2n-1$:

$$u_n(x,y) = f(x,y) + \sum_{i \in \mathcal{I}_{\Omega,\alpha}} \sum_{j=1}^{n_i} \sum_{k=1}^{n} w_{ijk} k(x,y,\xi_{ijk},\eta_{ij}) G\Big(\xi_{ijk},\eta_{ij},u_n(\xi_{ijk},\eta_{ij})\Big), \tag{2.5}$$

in which u_n is the new unknown solution and the approximate operator K_n is defined by:

$$(K_n G(u))(x,y) = \sum_{i \in \mathcal{I}_{\Omega,\alpha}} \sum_{j=1}^{n_i} \sum_{k=1}^{n} w_{ijk} k(x,y,\xi_{ijk},\eta_{ij}) G\Big(\xi_{ijk},\eta_{ij},u(\xi_{ijk},\eta_{ij})\Big).$$

For the practical purpose of solving (2.5), it is important to note that this relation is equivalent to solvability of the finite system

$$u_{vwz} = f(\xi_{vwz},\eta_{vw}) + \sum_{i \in \mathcal{I}_{\Omega,\alpha}} \sum_{j=1}^{n_i} \sum_{k=1}^{n} w_{ijk} k(\xi_{vwz},\eta_{vw},\xi_{ijk},\eta_{ij}) G(\xi_{ijk},\eta_{ij},u_{ijk}) \tag{2.6}$$

for the unknowns u_{vwz} for $v \in \mathcal{I}_{\Omega,\alpha}$ where $w = 1,\ldots,n_v, z = 1,\ldots,n$. There is a simple one-to-one correspondence between the solutions of (2.5) and (2.6) in which

$$u_n(\xi_{vwz},\eta_{vw}) = u_{vwz}$$

for $v \in \mathcal{I}_{\Omega,\alpha}, w = 1,\ldots,n_v, z = 1,\ldots,n$. (See e.g., [4] for more details).

Finally, in order to obtain the approximate solution of (1.1) in $N \sim mn^2$ nodes, where m is the number of sides which is equal to cardinal of $\mathcal{I}_{\Omega,\alpha}$, we end up the nonlinear system (2.6) with N equations. Availability of many well established iterative type methods for solving the nonlinear system such as Newton's method is a pronounced feature of the proposed method which obtains the unknown values.

We can summarize the proposed numerical algorithm over a polygon Ω with the vertices (α_i, β_i) for $i = 1,\ldots,l$, as follows:

The Algorithm

Step 1. Specify the parameters n, m and α. The integer n indicates that the exactness of the Gauss-like cubature will be $2n - 1$ and m being a number of sides of the polygon which are not orthogonal to the line $x = \alpha$ and not lying on it, where α is a fixed number belonging to the x coordinate including one from the edges of the polygon.

Step 2. Calculate the nodes (ξ_{ijk}, η_{ij}) and the weights w_{ijk} for $i \in \mathcal{I}_{\Omega,\alpha}, j = 1, \ldots, n_i, k = 1, \ldots, n$, respectively, by the relations (2.3) and (2.4).

Step 3. Constitute the system of equations with the unknowns u_n based on relation (2.5).

Step 4. Solve the nonlinear system (2.6) for the variables u_{vwz}, for $v \in \mathcal{I}_{\Omega,\alpha}$, $w = 1, \ldots, n_v, z = 1, \ldots, n$, which is equivalent to the approximate solution of the undetermined function $u(x, y)$ at the nodal points.

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3. Error analysis

Our analysis in this section rely on the framework developed in [12] for the projection methods.

For start, let us set $Tu \equiv KGu + f$, and $T_n u_n \equiv K_n Gu_n + f$, so the compact form of (2.1) and (2.2) is as follows, respectively:

$$u - Tu = 0 \tag{3.1}$$

and

$$u_n - T_n u_n = 0. \tag{3.2}$$

Assume that (3.1) has a solution $u_0 \in D$, where D is a subset of a Banach space X and T is a compact nonlinear operator from a domain D into X with continuous first derivative and second derivative bounded over $B(u_0, r) = \{u : \|u - u_0\| \leqslant r, r > 0\}$.

Following [13,17], let us suppose that the following assumptions are satisfied on the operator T_n:

(1) $\{T_n\}, n \geqslant 1$ is a collectively compact family on D.
(2) $T_n u \to Tu$, as $n \to \infty, \forall u \in D$.
(3) $\{T_n\}, n \geqslant 1$, possess continuous first and bounded second derivatives on $B(u_0, r)$.

To prove the unique solvability of the Eq. (3.2) in a neighborhood of an isolated solution u_0 of the Eq. (3.1), we need the invertibility of the linear operator $[I - T_n'(u_0)]$ which has been investigated in the following Lemma from [13]. Noting that the prime notation denotes the Frèchet derivative, while the linear integral operator $T'(u)$ is defined by

$$(T'(u)z)(x, y) = \int \int_\Omega k(x, y, s, t) \frac{\partial}{\partial u} G(s, t, u(s, t)) z(s, t) ds dt.$$

Lemma 2 (*From [13]*). *Assume that $[I - T'(u_0)]$ is nonsingular and the hypotheses (1)-(3) hold. Then the linear operator $[I - T_n'(u_0)]$ is nonsingular for sufficiently large n, say $n \geqslant n_1$, and*

$$\|[I - T_n'(u_0)]^{-1}\| \leqslant \beta < \infty, \quad n \geqslant n_1.$$

Now, using Theorem 3 in [14] that describes the conditions that are necessary to approximate solution of a class of nonlinear operator equations, we give the following main theorem together a bound for estimating the errors associated to n. Although, the presented proof follows very closely that of results of [12], we used the strategy related to the convergence verification of the solution (3.2) to the unique solution of (3.1) based on a general cubature formula for the projection methods as in Kaneko et al. in [12].

Theorem 1. *Let $u_0 \in D$ be an isolated solution of (3.1) and $\|K_n\| \leqslant \lambda$, where λ is a constant positive. Then Eq. (3.2) has a unique solution u_n in $\|u - u_0\| \leqslant \delta$ for some $\delta > 0$ and for sufficiently large n. Moreover, there is a constant $Q, 0 < Q < 1$ such that*

$$\sup_{\|u-u_0\|\leqslant\delta} \|(I - T_n'(u_0))^{-1}(T_n'(u) - T_n'(u_0))\| \leqslant Q$$

and

$$\|u_n - u_0\| \leqslant \frac{Q}{\lambda c \delta(1 - Q)} \|T_n(u_0) - T(u_0)\| \leqslant \frac{Q\|G(u_0)\|}{\lambda c \delta(1 - Q)} \|K_n - K\|.$$

Proof. For $\|u - u_0\| \leqslant \delta$ and $n \geqslant n_1$, we have

$$\|T'_n(u) - T'_n(u_0)\| = \|(K_nG)'(u) - (K_nG)'(u_0)\| = \|K_nG'(u) - K_nG'(u_0)\| \leqslant \|K_n\|\|G_u(u) - G_u(u_0)\| \leqslant \lambda c\|u - u_0\|$$

$$\leqslant \lambda c\delta. \tag{3.3}$$

From (3.3) we conclude:

$$\|(I - T'_n(u_0))^{-1}(T'_n(u) - T'_n(u_0))\| \leqslant \|(I - T'_n(u_0))^{-1}\|\lambda c\delta.$$

Let us set:

$$Q \equiv \lambda c\delta\|(I - T'_n(u_0))^{-1}\|,$$

here, we take δ so small, such that $0 < Q < 1$.

Now, the condition (2) of the mentioned hypothesis yields:

$$\exists n_2 : \ \forall \epsilon > 0, \quad n \geqslant n_2, \quad \|T_n(u_0) - T(u_0)\| < \epsilon.$$

Considering the Lemma 2, we can define α_n as follows:

$$\alpha_n \equiv \|(I - T'_n(u_0))^{-1}(T_n(u_0) - T(u_0))\| \leqslant \delta(1 - Q),$$

hence, if we consider $n \geqslant \max\{n_1, n_2\}$, then using the Theorem 3 in [14] we conclude that (3.2) has a unique solution in $\|u - u_0\| \leqslant \delta$, and the following inequality holds

$$\frac{\alpha_n}{1 + Q} \leqslant \|u_n - u_0\| \leqslant \frac{\alpha_n}{1 - Q}$$

and finally

$$\|u_n - u_0\| \leqslant \frac{\alpha_n}{1 - Q} = \left\|\frac{(I - T'_n(u_0))^{-1}(T_n(u_0) - T(u_0))}{1 - Q}\right\|$$

$$\leqslant \frac{Q}{\lambda c\delta(1 - Q)}\|T_n(u_0) - T(u_0)\|$$

$$\leqslant \frac{Q\|G(u_0)\|}{\lambda c\delta(1 - Q)}\|K_n - K\|.$$

This completes the proof. $\square$

4. Numerical illustrations and some experimental comments

In this section, we present some numerical examples for different polygonal domains and nonlinearities to clarify the accuracy of the proposed method. All the procedures contain the known parameters m, α and n, the vertices of polygon, the nodes and the weights of the univariate Gauss–Legendre quadratures over [-1, 1].

It should be noted that if α is a fixed number belonging to the x-coordinate including one of the edges of the polygon, the discretization nodes fall inside the polygon. On the other hand, if α is chosen such that it lies in the x-coordinate interval of the rectangle region enclosing polygon, then some of the nodes may be fall outside the polygon. Therefore the integrand function has to be computed also in the rectangular domain containing the polygon. Our numerical results demonstrate that the variation of α in $[a, b]$ does not much affect on the final solution, experimentally. However, the determination of an optimal α may be interest as a new research topic.

Table 1
Numerical results of Example 1 for different values of n and α with $m = 4$.

n	Nodes	Maximal error		
		$\alpha = 0.27$	$\alpha = 0.45$	$\alpha = 0.7$
2	24	1.0000×10^{-30}	1.0000×10^{-30}	1.0000×10^{-30}
3	48	4.9165×10^{-15}	5.0577×10^{-15}	5.1805×10^{-15}
4	80	7.3474×10^{-17}	7.4724×10^{-17}	7.5812×10^{-17}
5	120	7.3079×10^{-17}	7.3902×10^{-17}	7.4618×10^{-17}
6	168	7.6857×10^{-17}	7.7473×10^{-17}	7.8009×10^{-17}
7	224	7.7652×10^{-17}	7.8118×10^{-17}	7.8552×10^{-17}
8	288	3.0959×10^{-16}	3.1103×10^{-16}	3.1228×10^{-16}
9	360	1.5190×10^{-16}	–	–

Table 2
A comparison between the proposed algorithm and the numerical results of [15] for Example 2.

The proposed algorithm		Method in [15]	
Nodes	Maximal error	Nodes	Maximal error
8	1.7527×10^{-3}	4	6.124×10^{-2}
18	8.5358×10^{-5}	9	2.103×10^{-2}
50	1.2680×10^{-7}	25	5.930×10^{-3}
98	1.5442×10^{-10}	81	1.535×10^{-3}

Table 3
Numerical results of Example 3 for different values of n and α with $m = 2$.

n	Nodes	Maximal error		
		$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.15$
2	8	2.2082×10^{-10}	4.9323×10^{-11}	1.5270×10^{-10}
3	18	1.0540×10^{-13}	1.0153×10^{-14}	6.8218×10^{-14}
4	32	2.5420×10^{-17}	1.2200×10^{-18}	1.6060×10^{-17}
5	50	4.4645×10^{-21}	8.8553×10^{-22}	3.1018×10^{-21}
6	72	6.3278×10^{-23}	2.6294×10^{-23}	3.9919×10^{-23}
7	98	8.9730×10^{-22}	8.4254×10^{-22}	8.6931×10^{-22}

Table 4
Numerical results of Example 4 for different values of n and α with $m = 2$.

n	Nodes	Maximal error		
		$\alpha = 0.0001$	$\alpha = 0.1$	$\alpha = 0.18$
2	8	1.8382×10^{-6}	4.1396×10^{-8}	3.2427×10^{-7}
3	18	5.5528×10^{-10}	3.4380×10^{-12}	8.9580×10^{-11}
4	32	7.4767×10^{-14}	2.2984×10^{-16}	1.0135×10^{-14}
5	50	4.9496×10^{-17}	1.8681×10^{-17}	1.7496×10^{-17}
6	72	7.6546×10^{-19}	9.0600×10^{-20}	2.2380×10^{-19}
7	98	4.6017×10^{-17}	2.0901×10^{-17}	1.9090×10^{-17}

Table 5
Numerical results of Example 5 for different values of n and α with $m = 2$.

n	Nodes	Maximal error		
		$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 1$
2	12	1.0000×10^{-30}	1.0000×10^{-30}	1.0000×10^{-30}
3	24	1.0530×10^{-13}	1.1123×10^{-13}	1.0792×10^{-13}
4	40	1.1662×10^{-16}	1.2065×10^{-16}	1.1834×10^{-16}
5	60	7.8162×10^{-17}	7.9991×10^{-17}	7.8927×10^{-17}
6	84	1.3882×10^{-16}	1.4117×10^{-16}	1.3980×10^{-16}

Table 6
Numerical results of Example 6 for different values of n with $\alpha = 0.4$ and $m = 6$.

n	Nodes	Maximal error
2	36	1.0000×10^{-30}
3	72	4.8810×10^{-15}
4	120	1.5225×10^{-14}
5	180	9.5692×10^{-17}
6	252	9.9230×10^{-17}
7	336	9.1030×10^{-17}

The maximum absolute errors between the obtained approximate solutions and the exact solutions for nodal points in the relation (2.5), for different values of n and α have been tabulated in Tables 1–6. The parameter n shows that the accuracy of the used cubature is $2n - 1$, i.e., the cubature formula is exact over polygon for all bivariate polynomials of degree at most $2n - 1$. We also compute the number of nodes using the values of n and m such that m is a cardinal of $\mathcal{I}_{\Omega,\alpha}$ (according to the Lemma 1). It is shown that the maximum absolute errors are completely dependent on the values of the parameter n.
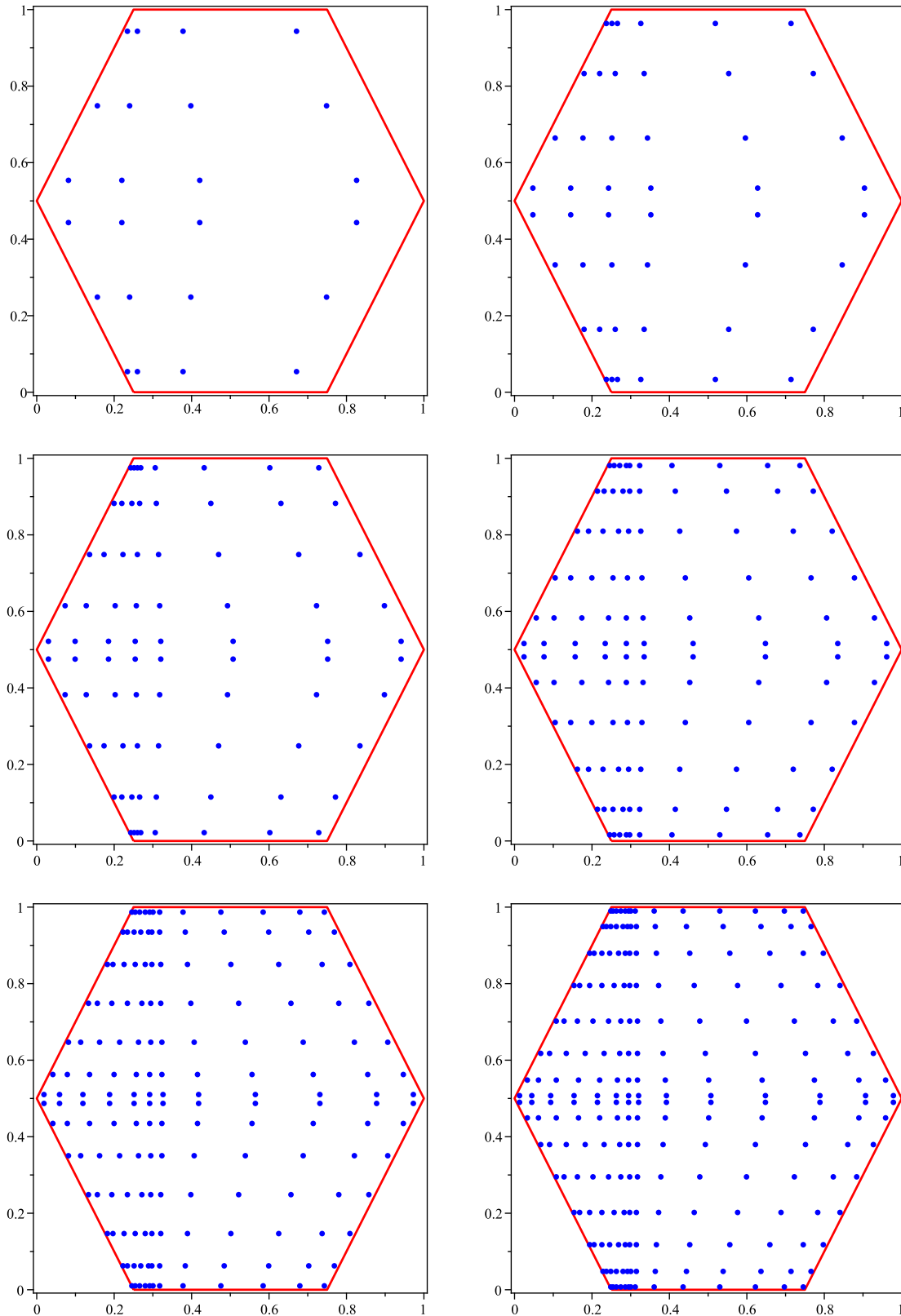
Fig. 1. Nodes distribution for $n = 2, 3, 4, 5, 6$ and 7 with $\alpha = 0.27$ in Example 1.

The numerical experiments indicate that the proposed approach offers a highly accurate approximate solution of the integral equations over polygonal domains and does not require the grid generation. We emphasize that in most considered cases, the small value of n (i.e., $n = 2$) gives a desired accuracy of solutions at the first stage of the algorithm.

All the computations were supported by Maple®, meanwhile in the process of algorithm, the nonlinear system of equation (2.6) can be solve by the *FSOLVE* command.

Example 1. Consider the following linear integral equation over a hexagon domain Ω

$$u(x,y) = y^2 - \frac{23}{192}x + \int\int_\Omega xsu(s,t)\,ds\,dt,$$

with the exact solution $u(x,y) = y^2$. The coordinates of the six vertices are $(0.25,0)$, $(0.75,0)$, $(1,0.5)$, $(0.75,1)$, $(0.25,1)$ and $(0,0.5)$.

The three values for α are chosen randomly from the interval $(0.25,0.75)$ as $\alpha = 0.27, 0.45$ and 0.7. Accordingly the parameter m is fixed as $m = 4$ and $\mathcal{I}_{\Omega,\alpha} = \{1,3,4,6\}$ based on the vertex (α_1,β_1) as $(0,0.5)$, and the number of nodes is also obtained based on $4n(n+1)$, for every n.

Numerical results corresponding to different values of n and α are listed in Table 1. As can be seen, the error is independent of the choice of the parameter α. For instance, in the case $n = 2$, which generates 24 nodes, the maximum error is quite low and the larger values of n gives a highly satisfactory results.

Fig. 1, exhibits the nodes distribution for different values of n in the case $\alpha = 0.27$. Here, the appropriate choice of α on the lower side of the hexagon Ω, causes all the nodes are placed inside the region. In order to validate the error behavior of the scheme, the graph of the absolute errors in this case is shown in Fig. 2.

Example 2 (*From [15]*).

$$u(x,y) = \frac{1}{(1+x+y)^2} - \frac{x}{6(1+y)} + \int\int_\Omega \frac{x}{1+y}(1+s+t)u^2(s,t)\,ds\,dt,$$

where Ω is the unit square region with the exact solution $u(x,y) = \frac{1}{(1+x+y)^2}$.

This problem has investigated numerically by the iterated spline discrete Galerkin method in [15]. In this method, a uniform partition is chosen on the unit square region together with n^2 grids and $(n+1)^2$ nodes where $n = 1,2,4,8,16$ and 32. Moreover, the resulting nonlinear algebraic systems have been solved by a Newton type method.

A comparison is made between the proposed method in this article and the results obtained in [15] which indicates that the results by our algorithm are better than those obtained by Carutasu [15], since the proposed algorithm needs only some discretization nodes instead of a mesh. It requires less computations to obtain the same order of errors. It is worth mentioning that the CPU times, which are very low for the proposed algorithm. Here, they range from 1.7 upto 10 s.
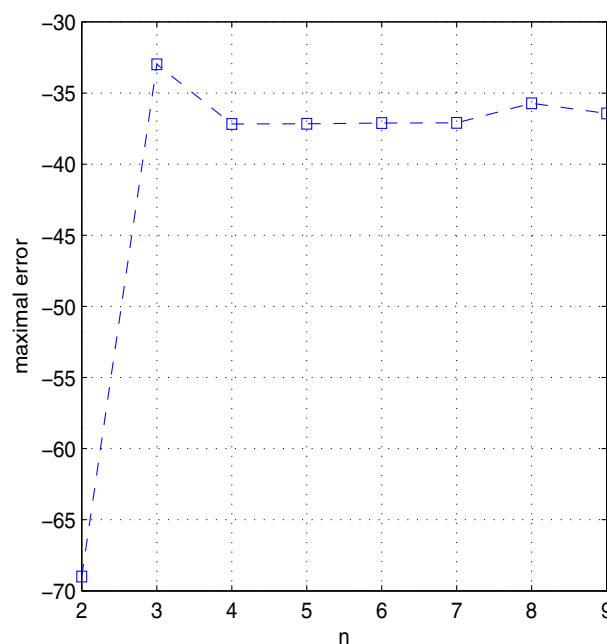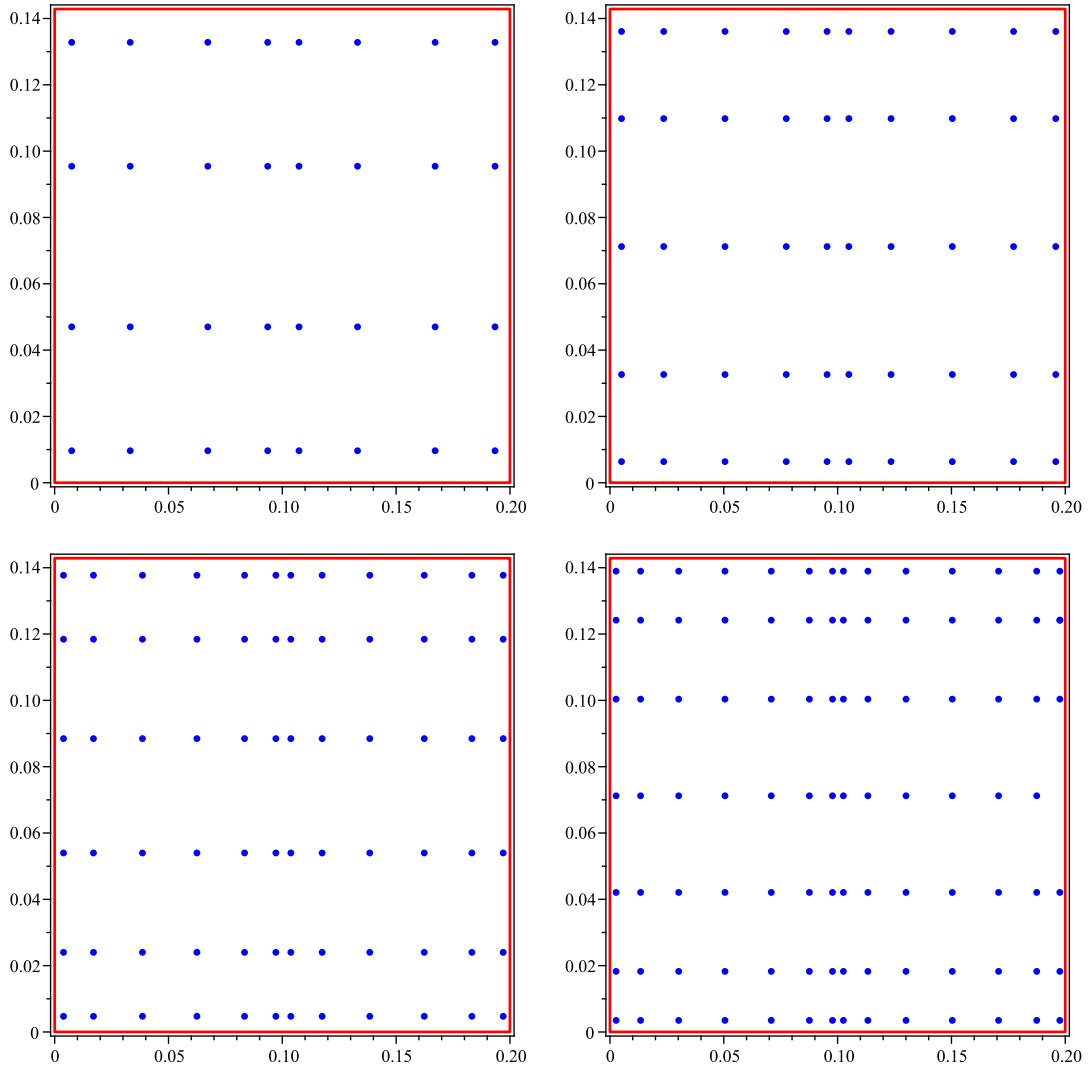

Fig. 2. Error behaviors in Example 1.

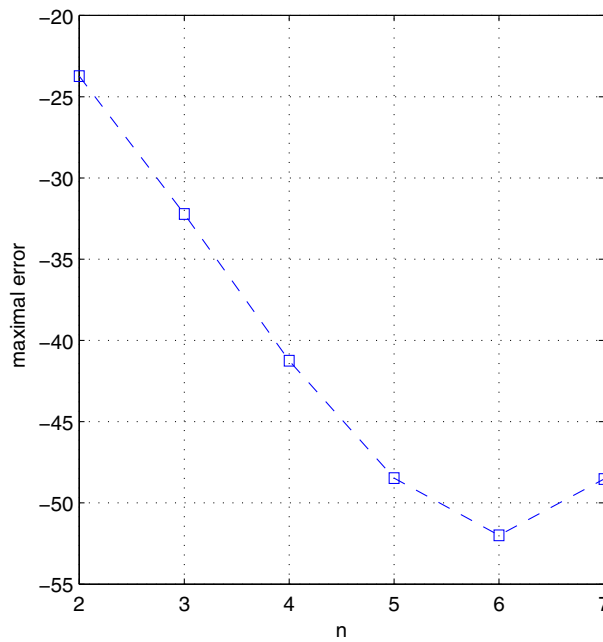**Fig. 3.** Nodes distribution for $n = 4, 5, 6$ and 7 with $\alpha = 0.1$ in Example 3.
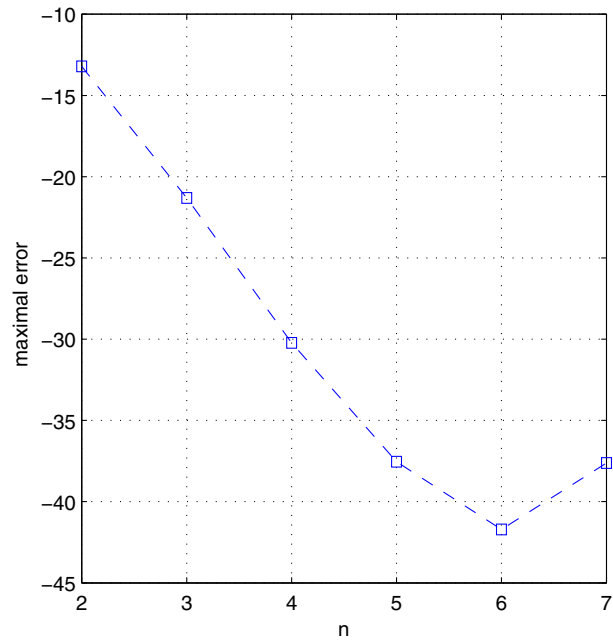

Fig. 4. Error behaviors in Example 3.
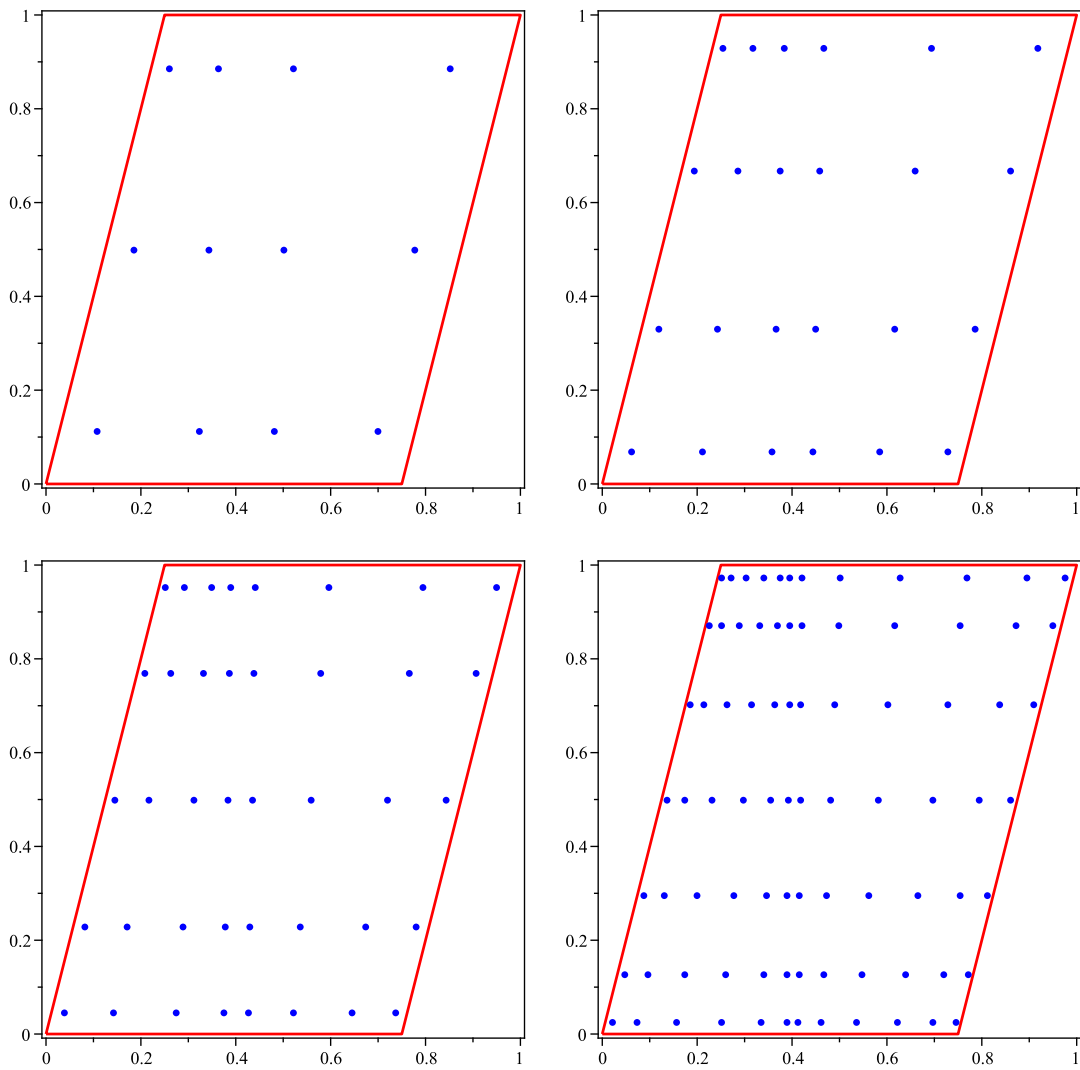
Fig. 5. Error behaviors in Example 4.

Fig. 6. Nodes distribution for $n = 2, 3, 4$ and 6 with $\alpha = 0.4$ in Example 5.

Example 3. As a third test problem, consider the following Hammerstein integral equation over a domain Ω, as a quadrangle which is defined by the four vertices $(0,0)$, $(1/5,0)$, $(1/5,1/7)$, $(0,1/7)$ and

$$u(x,y) = \sin x - \frac{45}{11173009}x + \int\int_\Omega xtu^3(s,t)dsdt,$$

such that $u(x,y) = \sin x$ is the exact solution.

Considering the vertex $(\alpha_1, \beta_1) = (0,0)$, gives $\mathcal{I}_{\Omega,\alpha} = \{2,4\}$, $m = 2$ and the number of nodes is achieved based on the formula $2n^2$, for every n. The related numerical results have been indicated in Table 3 and the nodes distribution as well as the error behavior for $\alpha = 0.1$ have been shown in Fig. 3 and Fig. 4.

Example 4. In this case, the following nonlinear integral equation has been considered

$$u(x,y) = \frac{3}{2}x - \frac{1453}{415340}\ln x + \int\int_\Omega s\ln xe^{u(s,t)}dsdt,$$

where its domain is as the same as previous example and the exact solution is $u(x,y) = \frac{3}{2}x$.

The numerical results are reported in Table 4. Here, as the parameter n increases, the absolute error decreases significantly and in the middle case $\alpha = 0.1$, the best results have been obtained. The error behavior is also shown for $\alpha = 0.0001$ in Fig. 5.

Example 5.

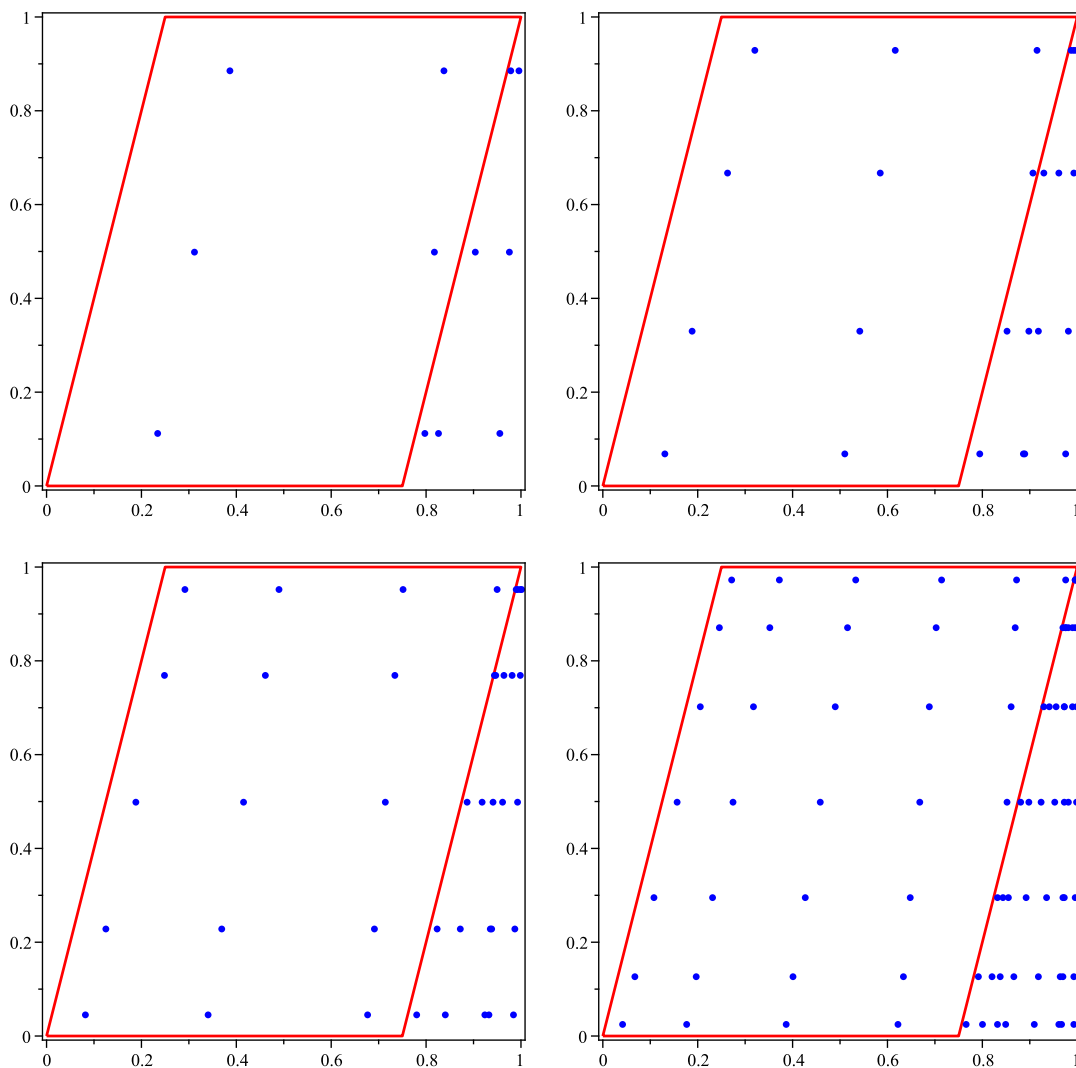$$u(x,y) = y - \frac{9}{64}\sqrt{x} + \int\int_\Omega \sqrt{x}su^2(s,t)dsdt,$$

Fig. 7. Nodes distribution for $n = 2, 3, 4$ and 6 with $\alpha = 1$ in Example 5.
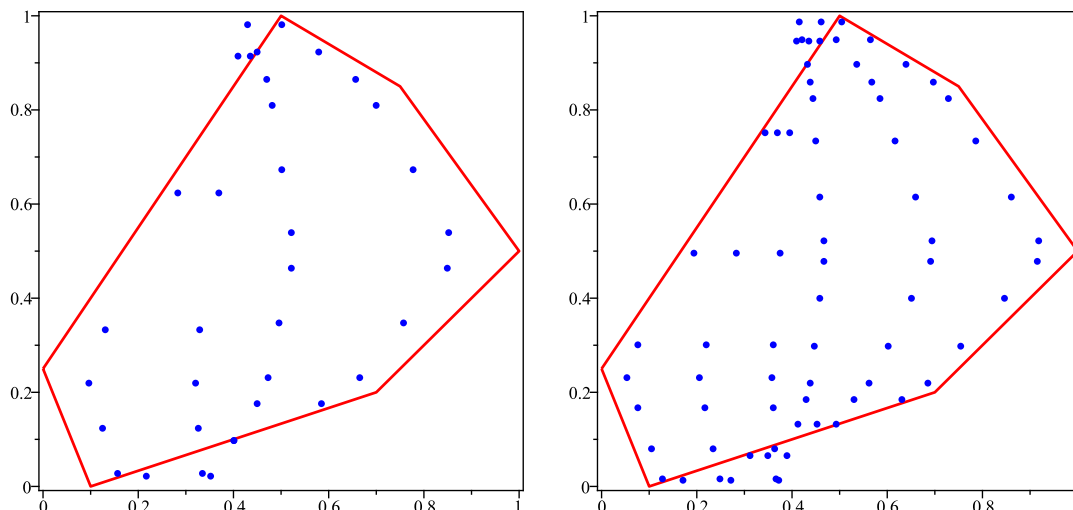
Fig. 8. Nodes distribution for $n = 2$ and 3 for $\alpha = 0.4$ in Example 6.

over a polygon Ω, which is a trapezoidal described by the vertices $(0,0), (0.75,0), (1,1), (0.25,1)$ and the exact solution is $u(x,y) = y$.

We work with the tow different values for α. The numerical results has been presented in Table 5. In the case $\alpha = 0.4$, all the nodes fall inside the trapezoidal as Fig. 6, however by choosing $\alpha = 1$ on the side of the rectangle enclosing trapezoidal, some of the nodes are set outside the integration domain as Fig. 7. It is notable that, if the integrand is continuous and computable also in the rectangle enclosing the trapezoidal, there is no significant differences between the use of the exterior or interior discretization nodes. However, clearly the use of interior points is preferable.

Example 6 (*From [1]*).

$$u(x,y) = x - \frac{24737}{160000}\sqrt{y} + \int\int_{\Omega} \sqrt{y}su(s,t)dsdt,$$

where Ω is a hexagonal region with the vertices $(0, 0.25), (0.1, 0), (0.7, 0.2), (1, 0.5), (0.75, 0.85)$ and $(0.5, 1)$. The exact solution is $u(x,y) = x$. Here, the integration region is not including the edge located in the *x*-axes, therefore choosing every α implies some of the nodes set out of the hexagon. We give the numerical results and the nodes distribution for the case $\alpha = 0.4$ in Table 6 and Fig. 8, respectively.

5. Conclusion

In this paper, we construct an approach based on a Gauss-like cubature to approximate solution of integral equations over a domain as polygon with piecewise smooth linear boundary in $\mathbb{R}^2$. The proposed scheme is a meshless like method which does not require the subdivision of the integration region. It needs only some discretization nodes instead of a mesh that reduces the computational complexity of the problem. It was shown that the proposed scheme is easy to implement, computationally attractive and offers a highly accurate approximate solutions over polygons.

References

[1] A. Sammariva, M. Vianello, Product Gauss cubature over polygons based on Green's integration formula, BIT Numer. Math. 47 (2007) 441–453.
[2] K.E. Atkinson, G. Chandler, The collocation method for solving the radiosity equation for unoccluded surfaces, J. Integral Equ. Appl. 10 (1998) 253–290.
[3] K.E. Atkinson, D. Chien, A fast matrix–vector multiplication method for solving the radiosity equation, Adv. Comput. Math. 12 (2000) 151–174.
[4] K.E. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, Cambridge University Press, New York, 1997.
[5] D.K. Kahaner, O.W. Rechard, TWODQD: an adaptive routine for two-dimensional integration, J. Comput. Appl. Math. 17 (1987) 215–234.
[6] K.E. Atkinson, A survey of boundray integral equation methods for the numerical solution of Laplace equation in three dimensions, Numerical Solution of Integral Equations, Plenum Press, New York, 1990. pp. 1–34.
[7] E. De Doncker, I. Robinson, An algorithm for automatic integration over a triangle using nonlinear exterapolation, ACM Trans. Math. Softw. 10 (1984) 1–16.
[8] S. Wandzura, H. Xiao, Symmetric quadrature rules on a triangle, Comput. Math. Appl. 45 (2003) 1829–1840.
[9] A. Sammariva, M. Vianello, Meshless cubature by Green's formula, Appl. Math. Comput. 183 (2006) 1098–1107.
[10] A. Sammariva, M. Vianello, Meshless cubature over the disk by thin plate splines, Comput. Appl. Math. 221 (2008) 430–436.
[11] C.J. Li, C. Dagnino, An adaptive numerical integration algorithm for polygons, Appl. Numer. Math. 60 (2010) 165–175.
[12] H. Kaneko, R.D. Noren, Y. Xu, Numerical solutions for weakly singular Hammerstein equations and their superconvergence, J. Integral Equ. Appl. 4 (1992) 391–407.
[13] R. Weiss, On the approximation of fixed points of nonlinear compact operators, SIAM J. Numer. Anal. 11 (1974) 550–553.

[14] G. Vainikko, Perturbed Galerkin method and general theory of approximate methods for nonlinear equations, USSR Comput. Math. Math. Phys. 7 (1967) 1–41.
[15] V. Carutasu, Numerical solution of two-dimensional nonlinear Fredholm integral equations of the second kind by spline functions, Gen. Math. 9 (2001) 29–43.
[16] D.J. Allman, A quadrilateral finite element including vertex rotations for plane elasticity analysis, Int. J. Numer. Methods Eng. 26 (1988) 717–730.
[17] K.E. Atkinson, The numerical evaluation of fixed points for completely continuous operators, SIAM J. Numer. Anal. 10 (1973) 799–807.